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MATHEMATICAL THINKING: THE STRUGGLE FOR MEANING

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This paper argues that mathematical thinking is *not* thinking about the subject matter of mathematics but a style of thinking that is a function of particular operations, processes, and dynamics recognizably mathematical. It further suggests that because mathematical thinking becomes confused with thinking about mathematics, there has been little success in separating process from content in the classroom presentation of the subject. A descriptive model of mathematical thinking is presented and then used to provide a practical response to the questions, Can mathematical thinking be taught? In what ways? The teacher is encouraged to recognize both what constitutes mathematical thinking, whether in the mathematics class or some other, and what conditions are necessary to foster it.

Most schools assume that by teaching mathematics compulsorily and over a number of years they are providing the conditions through which pupils will develop their mathematical thinking. This assumption, usually unchallenged, rests on a view of mathematics as a logically developed discipline, together with the expectation that the logic will spill over and be absorbed by the pupils into all aspects of their lives as they pursue a study of the content of mathematics, for example, in learning number, geometry, trigonometry, or algebra. Experience, however, tells a very different story. Few pupils leave the school system with mathematical success as measured by examinations, and those who do consistently surprise their university tutors by their *lack* of facility in thinking mathematically. Many mathematics educators articulate the problem in terms of content versus process. Certainly an inordinate amount of time in schools is spent teaching mathematical content and techniques while the process, the means through which mathematics is derived, receives little attention. But even where attempts have been made to introduce an emphasis on process into the curriculum, little impact is visible. The reason is partly that once process is enshrined in texts, it becomes content. But, also, exploring process is not very profitable when teachers do not understand the kinds of thinking from which the process springs.

In this paper I propose addressing the following questions:

- What is mathematical thinking?
- What does it have to do with mathematical content?
- Can it be taught?

First, a model of mathematical thinking is outlined in terms of operations, processes, and dynamics. Researchable questions thrown up by such a model are identified. Next, an example is used to explore the distinction between an approach to mathematics dictated by the model and a conventional "mathematical" presentation. Finally, the research basis of the model is used to justify encouraging teachers to adopt this approach in order to stimulate and use their pupils' mathematical thinking.

The following axiom underpins the approach:

> *Thinking is the means used by humans to improve their understanding of, and exert some control over, their environment.*

It follows from the axiom that as individuals increase their awareness of their thinking "field," so they extend their range of possible choices.

WHAT IS MATHEMATICAL THINKING?

I draw a clear distinction between mathematical thinking and the body of knowledge (i.e., content and techniques) described as mathematics. The style of thinking labeled mathematical is pertinent whatever the content to which it is being applied. It is mathematical not because it is thinking about mathematics but because the operations on which it relies are mathematical operations. Its field of application is general. Like the scientific method, which does not necessarily pertain to science alone, mathematical thinking is used when tackling appropriate problems in *any* context area, although questions of a mathematical nature might more readily expose such thinking. A problem is appropriate to mathematical thinking when it provokes or responds to the use of the components identified below. It has been argued that mathematical thinking is the means by which infants first organize the information they gather through their senses in order to learn from their environment and, in particular, in order to learn to speak (Gattegno, 1973).

If thinking is a way of improving understanding and extending control over the environment, mathematical thinking uses particular means to do this, means that can be recognized as arising from or pertaining to the study of mathematics. These means will be described as the operations, processes, and dynamics of mathematical thinking.

The Operations of Mathematical Thinking

What do we think about? An idea, an observation, a happening—any event can provide a stimulus to begin thinking. Such events are the elements on which mathematical thinking operates. The thinking requires that elements be acted on in some way, and the methods, or operations, used are all identifiably mathematical (Figure 1). For example, when faced with a group of objects, a child might think about how many. The mathematical nature of this thinking would be recognized by all teachers as *enumeration*. However, just as mathematical is the thinking necessary for repetition, or *iteration*, since it is dependent on pattern recognition and continuation. Repetition can often be used to great effect, for example, in Bach's preludes or in the drawings of Escher (Hofstadter, 1979).

The study of relationships is central to doing mathematics. Elements can be related in many different ways to themselves or to other elements, for exam-

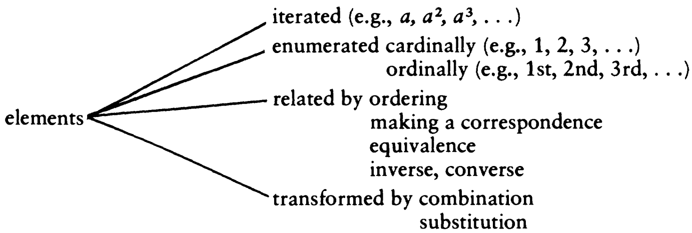

Figure 1. Mathematical thinking as operations on elements.

ple, by *ordering, making correspondences,* or *forming equivalence classes.* Think of "Goldilocks and the Three Bears," in which ordering (Daddy Bear, Mummy Bear, and Baby Bear), correspondences (bears matched to big bed, middle-sized bed, and baby bed, etc.), and equivalence classes (Daddy Bear has the big bowl of porridge, the big bowl is by the big chair, so Daddy Bear has the big chair, etc.) are central to the story.

Combining elements or *substituting* one for another can transform them from their current state into a new state. Indeed, many of the plots of Shakespeare's comedies depend on just such transformations. The examples above provide justification for the earlier assertion that mathematical thinking is independent of the content area in which it is being applied. The operations of mathematical thinking are necessary for understanding and using the ideas.

More recognizable mathematical operations are, of course, pertinent to mathematical thinking. When weighing ingredients to make a cake, for example, the cook *adds* and *subtracts* quantities before arriving at a satisfactory amount. Most teachers are happy to acknowledge the mathematical nature of these actions. Yet, the task demands the operations of mathematical thought as well as the mathematical actions. A transformation must be made from a current state (too much flour, for example) to a new state (the correct amount). It is effected by subtracting, which can be effectively used because it is supported by the structure of thought that permits continuous transformation.

Small-scale observations of children from nursery age (3 years) upward have yielded evidence of their use of mathematical thinking operations. The next step is to codify these observations, identifying separately that behavior resulting from mathematical thinking and that resulting from the application of mathematical knowledge or skills. For example, evidence of a very young child's ordering, on the one hand, would be taken as an action resulting from mathematical thinking. Counting, on the other hand, would depend on familiarity with the use of numbers.

The Processes of Mathematical Thinking

Four processes can be shown to be central to mathematical activity and yet, as before, to have general application. The four processes are (a) specializing, (b) conjecturing, (c) generalizing, and (d) convincing.

Specializing. When one is faced with a question or problem, a powerful way to explore its meaning is by examining particular examples. Such *specializing* is the key to an inductive approach to learning and is observed as natural to the learning of children. Each example provides the opportunity for manipulating elements that are concrete in the child's thinking, whether they are physical manifestations or ideas.

Conjecturing. When enough such examples have been examined, *conjecturing* about the relationship that connects them happens almost automatically. Through conjecturing, a sense of any underlying pattern is explored, expressed, and then substantiated.

Generalizing. The recognition of pattern or regularity provokes the statement of a *generalization.* Such statements appear to be the building blocks used by learners to create order and meaning out of an overwhelming quantity of sense data, and it is on such generalizations that much behavior depends.

Convincing. To become robust, a generalization must be tested until it is *convincing.* First the thinker convinces himself or herself and then the world outside. The convincing process is the means by which a generalization moves from being personal to being public. A picture of the deductive approach is obtained by inverting the order of the processes. Beginning with a generalization, one explores the web of conjectures it provokes and tests them against particular specializations.

Inductive learning: SPECIALIZING → CONJECTURING → GENERALIZING

Deductive learning: GENERALIZING → CONJECTURING → SPECIALIZING

Convincing, in both the inductive and the deductive cases, is not simply a matter of verification. To underline this claim, it is valuable to introduce the notion of a monitor, derived from the work of Schoenfeld (1983). The monitor represents an internal enemy who pushes past complacent acceptance, doubting and probing an argument, querying assumptions, and negotiating meanings toward the best possible proof in the circumstances. The notion of proof is again a mathematical notion and one that distinguishes mathematical from scientific activity. A proof is an argument deduced from a set of axioms and independent of empirical trials. For a proof to be acceptable the "logic" of its deduction must convince an external "enemy," usually the community at which it is aimed. Proofs can thus be seen as attempts at constructing convincing arguments; they are neither universal nor final. What a proof does and does not prove can, indeed should, be investigated (Bloor, 1976; Lakatos, 1976).

Two areas for investigation are present here. One could be labeled "What does a proof prove?" Although clearly derivable from the work of Lakatos, it can be fruitfully explored with young learners. The approach "convince me" is both more open to use with younger children and more attractive to them than the rigorously mathematical proving process often derived without earlier experience of this type. The effects on their later adoption of the proving process should be researched. The second area is the amenability of young children to the proving process, which has been investigated by Bell (1976) and Balacheff (1981) and which requires further attention from researchers.

The Dynamics of Mathematical Thinking

A helical extension of the framework originally offered by Bruner, Goodnow, and Austin (1956) has been proposed by Mason (Mason, Burton, & Stacey, 1982). In such a representation, the dynamics of mathematical thinking are displayed by movement around or between an unspecified number of loops, each new loop building on the understandings and awarenesses achieved in traversing previous loops (see Figure 2).

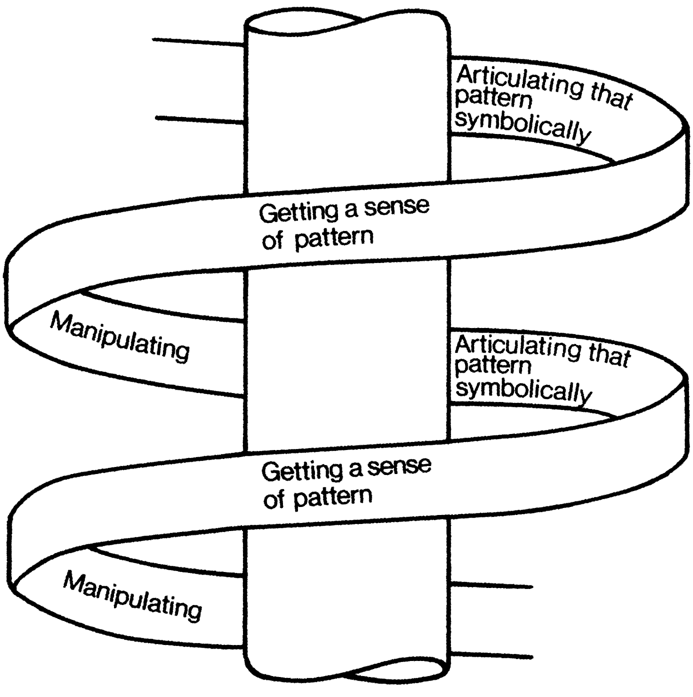

Figure 2. The dynamics of mathematical thinking. From *Thinking Mathematically* (p. 181) by J. Mason, L. Burton, and K. Stacey, 1982, London: Addison-Wesley. Copyright 1982 by Addison-Wesley Publishers Limited. Reprinted by permission.

The process is initiated by encountering an element with enough surprise or curiosity to impel exploration of it by *manipulating*. The element may be a physical object, a diagram, an idea, or a symbol, but it must be encountered at a level that is concrete, confidence inspiring, and amenable to interpretation. A perceived gap between what is expected from the manipulation and what actually happens provokes tension that provides a force to keep the process going until some *sense of pattern* or connectedness releases the tension into achievement, wonder, pleasure, or further surprise or curiosity that drives the process on. Although the sense of what is happening remains vague, further manipulating is required until the sense can be expressed in an *articulation*.

Articulations do not have to be verbal. They might well be in concrete, diagrammatic, or symbolic form, but they will crystallize the essence underlying the sense achieved as a result of the manipulations. An achieved articulation immediately becomes available for new manipulating; hence the wraparound of the helix. Each successive loop therefore assumes that the thinking is more complex, since the new elements being manipulated are the achieved articulations of the previous loop. This complexity might be obtained from increasing generality or from increasing refinements. The connectedness of the loops always permits the thinker to have the opportunity to track back to previous levels and recreate articulations that might have become unstable. The new is mastered by reference back to what has been previously mastered either by demanding particular instances (specializing again) or by providing ones that test against an articulation. The process is going on continually—no doubt you used it when reading the paragraph above by relating specific instances from your own experience to see how they fit the generalities being expressed.

The development of ideas of number in children provides a good example. Early in life, with growing muscle coordination, the child manipulates objects with increasing confidence. There follows a developing sense of oneness, twoness, and so on, together with a feeling for the idea of matching, and later, one-one matching. After much exposure, oneness and twoness become associated with the words *one* and *two*, and later still, a written form of 1 and 2 are recognized. Long before the written symbols are concrete, the verbal forms begin to be combined and used in thinking about, for example, situations of more or less than (e.g., Have we more plates than people? Have we enough cups?) Later still, verbal articulations become concretely manipulable as symbolic statements, such as 7 is 3 more than 4. These are the foundations of arithmetic, but articulation in written form, $7 = 3 + 4$, must again wait for the development of an underlying grasp or sense of what is meant, preferably through the verbal constructions that are becoming part of the child's automatic, meaningful language. The cycle continues through multiplication and division, fractions and decimals, negative numbers, square roots, powers, logarithms, sines and cosines, and so on. In parallel, the possibility of representing a range of values by letters becomes increasingly concrete, and the shift to generalizing that is represented mathematically by algebraic activities is supported by images such as graphs and coordinates and through the experience of number games like Think of a Number. The cycle continues into the highest realms of number abstraction. At any point, if faced with a lapse of understanding, confusion, or total bafflement, the sensible action is to backtrack down the helix, appealing to a sense of pattern achieved from further concrete examples. Manipulating particular examples, that is, specializing, with an eye to discerning meaning at the point of difficulty, is then an attempt to grip the helix firmly and climb back up on more reliable foundations.

Although manipulating, getting a sense of pattern, and articulating describe the cognitive activities propelling mathematical thinking, the affective components also demand attention. The ebb and flow at the cognitive level is charted by affective responses that can be observed as passing through three phases: *entry, attack,* and *review.*

As meanings are sought, commitment is tentatively aroused. This phase of engaging is described as *entry.* Surprise, curiosity, or tension creates an affective need. To resolve this need requires further exploration that, in turn, satisfies the cognitive need to get a sense of the underlying pattern. One explanation of this struggle for meaning describes it in terms of a basic human need to resolve "cognitive conflict" (Bruner et al., 1966). However, there are two possible affective means of dealing with such conflict. One is to engage further and *attack* the cause of the conflict. The other is to withdraw with a sense of failure and incapacity. Moving from the entry phase to attack and engaging further is likely only in a person who is already aware of enough success from previous attacks for his or her confidence to cope with the possibility of failure on this occasion. It is not cognitive input so much as the feeling accompanying that input that dictates whether thinking engages or subsides. This interdependence of the cognitive and the affective is consequently central to learning. A sense of achievement that accompanies an articulation provides the momentum to look back and check the achieved generality against the original state and the experience in attack and to look forward from the achieved generality toward further questions that it provokes. This opportunity for reflection and extension is described as *review* (see Figure 3).

The helix provides a representation of mathematical thinking that underlines the interconnectedness of cognition and emotion. Although they are interlocked, an observer can distinguish distinct behavior and emotional states. Confidence results from manipulating elements that are concrete to the thinker. It provides the momentum to move from entry to attack and to develop a sense of pattern out of the concrete specializations. Curiosity and tension sustain the attack to the point where an articulation is possible. The satisfaction of achievement fuels a review and the further need to place the achieved understanding in a wider context. Thus, the cycle continues. Blocks,

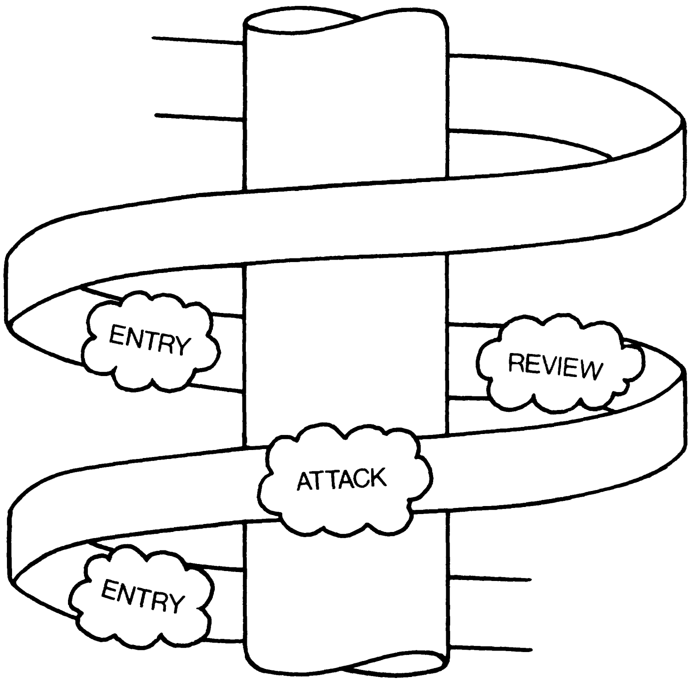

Figure 3. Affective phases in mathematical thinking.

misunderstandings, or conjectures that fail under testing cause an oscillation within loops and across loops. The apparent linearity of the representation is not observed in practice, therefore.

Again, considerable work remains to be done in identifying learner responses appropriate to each phase of the loop. What kind of articulation can a teacher expect to hear from a child who has a well-established sense of counting numbers, say, or at a higher level, of the meaning of the trigonometric functions? What behavior, verbal and nonverbal, indicates that the learner is still trying to establish that sense and requires further experience?

WHAT DOES MATHEMATICAL THINKING HAVE TO DO WITH MATHEMATICS?

The following question is the kind that might provoke both mathematical thinking and mathematics:

At a warehouse I was informed that I would obtain 20% discount on my purchase but would have to pay 15% sales tax. Which would be better for me to have calculated first, discount or tax? (Cf. Mason et al., 1982, p. 1).

The following is an annotated record of a person's response to this question. On hearing the question, the person made an entry conjecture:

I would expect to benefit from obtaining discount first, then adding the tax to a smaller price.

It was followed by a specialization:

I'll try it for an item priced £100. Discount first means subtracting 20% of £100, which gives £80. Adding 15% tax, which is £12, gives £92 as the final price. The other way round is tax first; 15% of £100 added on is £115. Now 20% discount means subtracting £23. Result, £92. That's not what I expected! The order of calculation seems to make no difference.

Surprise fueled attack and a further question leading to another specialization.

I wonder if that is so for a differently priced article? Say £65. Tax first this time, £78. Now discount, £62.40. Discount first, £52. Then add tax, £62.40. Aha! So it makes no difference.

A confirmed conjecture leads to the need for convincing.

Now how to show that for any price. If the article costs £A: Tax first means £A + 15%A = 115%A.
Less discount leaves 80% of 115%A.
Discount first gives 80%.
Tax next 115% of 80%A.
Now, multiplication is unaffected by order, so these are the same.

Finally, review provokes a new question and would lead to another entry.

Oops! Wait a minute. I've only shown that the order of calculation makes no difference if the discount is 20% and the tax 15%. Would it still make no difference if the discount were 17% and the tax 8%, or indeed, anything else?

The example displays all four processes of mathematical thinking and links them to the helix. In order to specialize, the person manipulated particular elements (numbers) that were concrete for her in order to get a sense of what was going on and to generate the articulation of a generalization. The entry was tentative, but surprise quickly fueled an attack on the question that culminated in a resolution that displayed generality. She used the tools of algebra to help convince herself that the generalization was robust.

The processes by which this resolution was obtained, made more obvious by the use of the helix of mathematical thinking, were independent of the mathematical content. The mathematics demanded by the question was some elementary manipulation of numbers, particularly percentages, followed by the use of algebra.

Algebra was a powerful tool for convincing in the example because of the kind of question, but in an example not overtly mathematical other tools would also have the power to convince. If you have ever tried to shift a heavy item of furniture around a corner, you will be aware of different means to convince yourself of the conditions under which it can or cannot be done, including measuring, diagramming, and modeling.

The resolution also demonstrates why mathematical thinking does not automatically emerge from a study of mathematics. A model answer would have suppressed all evidence of mathematical thinking and presented only the abstract application of the algebraic tools. Further, such a formal treatment removes examples of the negotiation of meaning (e.g., through specializing), of the recognition of constraining factors (e.g., that generality of price does not automatically imply generality of percentage changes in price), and, in particular, of the feelings that were evoked. The mathematics is presented as a closed manipulation of techniques, whereas the mathematical thinking demonstrates open inquiry. An over-conscientious concentration on the content of mathematics would therefore be expected to obstruct the development of the kind of awareness on which mathematical thinking is based.

Every example calls out some tools of mathematical thinking *and* of mathematics. A choice of such examples, and the opportunity to explore them, provides a rich display of mathematical thinking as well as a justification for the use of mathematics. Therefore I subscribe to the assertion that "the reason why we should study mathematics is because it educates . . . a third eye . . . capable of scrutinizing relationships *per se* . . . and capable of indefinite extension" (Gattegno, 1963, p. 98).

CAN MATHEMATICAL THINKING BE TAUGHT? A PROCESS OF ENRICHMENT

The discussion above has underlined the relationship between mathematical thinking and learning, and between learning and expanding awareness. If mathematical thinking is a natural means by which we classify, combine, relate, and transform information, then children bring an experience of these mathematical thinking operations with them when first they come to school, and such experience is available if teachers know of it and how to use it. Learning is then not simply a function of input, or practice, but depends on a conscious reflection on, at the same time, what is being done and why. The quality of the pupils' mathematical thinking and more especially of their responses to mathematics can be affected by their experiences. Not surprisingly, the quality can deteriorate as well as be enriched if only because of the interlocking of cognitive and affective experience (Buxton, 1981).

A number of studies have been undertaken in the United Kingdom using the above framework (Burton, 1980b; see also the 1982 Open University course EM235 *Developing Mathematical Thinking*). The major difference between these studies and the usual approach to generating problem-solving processes in the classroom has been their emphasis on the need to make the processes overt and to concentrate on them so that they become the focus of the learner's attention and their power to inform and direct an inquiry can be recognized.

The Skills and Procedures of Mathematical Problem Solving Project (Burton, 1980b) concentrated on children between the ages of 9 and 13 years.

Although the children were of mixed abilities, both their confidence and their capacity consciously to scan and choose an appropriate strategy were positively affected by their participation in the project. Particularly noticeable was the degree to which they adopted a wide range of techniques for representing problem information as part of their entry and attack behavior. Abundant evidence was available of the role of such representations in providing a necessary basis for manipulation toward a sense of pattern in the children's exploration of the problems (in this respect see, e.g., Burton, 1980a). Such studies support a philosophy and methodology of teaching quite distinct from those normally observed. Certainly the analysis indicates an approach that might be termed "child-centered," but it is not undertaken to see how the child fits a predetermined model of development or displays expected behavior. Putting children at the center of their own learning accepts that they come to the classroom equipped with a vast array of tools that have already served them well and that they can continue to use and refine. Respect for the power of those tools leads to the derivation of a curriculum that requires their application. Thus, since the teacher knows that the child is already an accomplished classifier, it makes sense to let him or her build confidently on that skill and to provide the opportunities so to do. Equally, since the child has effectively used the ordering operation in preschool learning, it is consistent to use that operation in the kinds of experiences offered in school. But it is only as one becomes aware of one's thinking tools that one will exercise their power. As long as they are not brought to the conscious level, examined, and discussed, their application will remain unrecognized. Offering overt opportunities for specializing, generalizing, conjecturing, and convincing enables the thinker to encounter aspects of his or her own thinking more deeply. Being aware of the operations of mathematical thinking helps both teachers and pupils to recognize their own power in thinking about mathematical experiences. A sensitive appreciation of the dynamics of mathematical thinking draws attention to the decisive role of feelings in thinking and to the need to interlock action with thought and expression.

Capturing the Feelings in Words

Pupils need tools to help them structure their responses so that they can build their reflective powers. Further, they need encouragement to capture their feelings at the moment of expression. Consequently, students of all ages have been encouraged to develop the use of particular words that reflect their responses as they tackle questions (Floyd, 1982; Mason et al., 1982). These words can then act as triggers to further thought as well as providing mental markers. The most powerful of such words are STUCK! and AHA! The action of writing STUCK! seems to release the energy that has been blocked by the state of being stuck. Instead of being a plaintive cry for help, it jogs the thinker into the use of the processes of specializing, conjecturing, generalizing, and convincing. Associated with STUCK! are questions such as "What do I KNOW?" "What do I WANT?" and "What can I INTRODUCE?" which are all appropriate to the type of activity typical of entry. During the attack phase, STUCK! calls out TRY, MAYBE, BUT WHY? When reviewing what one has done, the words suggested are CHECK, REFLECT, and EXTEND. Without in any way insisting on a particular choice of words—indeed, the more personal the choice the more likely they are to prove useful—Mason and his colleagues have found that the action of *writing* such annotations both facilitates results and stimulates an awareness of mathematical thinking. Introducing the writing of such words as STUCK! AHA! REFLECT! as a natural part of classroom activity seems consistent with the purpose of the classroom. In particular, it emphasizes that getting stuck is a natural phenomenon that is conducive to further learning. The derivation of the means by which "STUCK!" can be undone and turned into "AHA!" fixes the experience in one's consciousness, enriched by a strong sense of accomplishment and confidence.

Developing a Questioning Atmosphere

"The answer is 42. What was the question?" comes from a satirical radio and television program on the British Broadcasting Corporation called "A Hitchhiker's Guide to the Galaxy." It is easy to draw an analogy with much of what is presented to students in formal classrooms. Small children have no shortage of questions of their own, but the formality of the curriculum rapidly represses them. Pupils end up instead with curriculum answers to questions that they do not possess. As demonstrated in the sample resolution above, emphasizing the personal enrichment of asking questions and then examining the implications of such question asking can become a natural way of exposing students' mathematical thinking. They can notice and wonder at the unexpected and the changing, challenge the hidden assumptions in the commonplace and accepted. Most of all, they can conjecture explanations for what they observe, learn to test their conjectures, and reflect on what they have uncovered. One of the great advantages of the classroom is that it can provide a group experience in which conflicting conjectures articulated by different members of the group can create the spark necessary to shift the thinking process from *entry* to *attack*. In a mutually supportive atmosphere, the awareness of a gap that opens between where I think I am and where others appear to be—between what I perceive compared with others—is one of the ways in which tension can arise and mathematical thinking can be generated. Such gaps do occur inside individuals but can more frequently be induced when a group works together. Where the gap is knowledge based, insecurity and panic can develop. Where the gap represents the distance between different conjectures, the possibility exists for the testing of those conjectures, for example by specializing, and for the negotiation of meaning along the way. Hence the power of such techniques as synectics (Gordon, 1961) and CoRT (de Bono, 1976) has been demonstrated when they are applied to knowledge-based learning (e.g., Edwards & Baldauf, 1982), and to creativity in, for example, the use of "brainstorming" by the Creative Studies Course at the State University of New York at Buffalo (Parnes & Noller, 1972–1973).

An Example of the Approach

Richard, aged four, carries his father's briefcase downstairs each evening in preparation for the next day. One evening, his mother puts a large quantity of coins into the briefcase for his father to bank the following day. Richard is unable to lift the briefcase. This, for Richard, is a problem that provokes investigation. When presented with this scenario, teachers in training respond by saying, "Explain to Richard that there is something heavy in the briefcase" or "Show Richard the heavy coins in the briefcase."

Here is an alternative approach. First, pose the problem. Well, Richard, *what has changed?* Now, conjecture:

- Perhaps Richard has changed, that is, he is no longer strong enough to lift the briefcase.
- Perhaps the conditions surrounding the briefcase have changed, that is, the briefcase has become glued to the floor.
- Perhaps the briefcase itself has changed, that is, it is no longer the same briefcase, or something about it is no longer the same.

Next, test each conjecture:

- Is Richard feeling ill?
- Is the floor different?
- Has Richard's father changed his briefcase?

What remains? Something different about the briefcase. Let us then examine the briefcase and its contents, starting with the briefcase empty, refilling it item by item, and testing each time. What does Richard find out?

1. He can investigate his problem.
2. He can conjecture and test his conjectures.
3. He can construct an argument step by step.
4. His curiosity can be fed in different ways.
5. He can create his own resolution of the problem.
6. Heaviness has meaning because of the process Richard undergoes to establish that meaning.

The most gentle explanation, the most sensitive "showing," cannot encourage Richard's mathematical thinking, and it kills his problem stone dead!

CONCLUSION

The key to recognizing and using mathematical thinking lies in creating an atmosphere that builds confidence to question, challenge, and reflect. Behind such behavior is an acknowledgment of the need to:

- query assumptions
- negotiate meanings
- pose questions
- make conjectures
- search for justifying and falsifying arguments that convince
- check, modify, alter
- be self-critical
- be aware of different approaches
- be willing to shift, renegotiate, change direction

Relevant teacher behavior to establish and maintain this kind of classroom atmosphere is reflected in the use of questions such as:

- Why do you think that?
- What do you notice?
- Is there another way?
- What if . . .?
- Can you convince a friend?
- Can you find a counterexample?

Such experiences in the classroom can only increase the pupil's awareness. But awareness means more than that. It is the bridge that connects disparate areas of knowledge, information, experience, techniques, perception, and feeling to each other and to the world outside. And awareness operates on itself. By becoming aware of the existence and use of mathematical thinking, the child shifts content learning to the object rather than the subject of the curriculum. Increased awareness does not just happen: it must be fostered, tendered, and built on in a conscious way. The struggle for meaning is pursued, therefore, in an atmosphere supportive of the mathematical thinking that will mediate that struggle. This requires a recognition by both teachers and those who are taught

- of the operations of mathematical thinking, which provide the tactics of the struggle;
- of specializing, generalizing, conjecturing, and convincing, which are the processes of the struggle;
- of the helix of mathematical thinking, which links the cognitive and the affective in the dynamics of the struggle.

Accepting the implications of this model leads teachers and taught into a different relationship that not only encourages the development and use of mathematical thinking but gives direction to mathematics education. Systematic observation and recording of such changes by researchers will, I hope, provide evidence of the enrichment the changes bring to pupils' experience as well as support attempts to convince teachers of the validity of the approach.

REFERENCES

Balacheff, N. (1981). Une approche expérimentale pour l'étude des processus de résolution de problèmes [An experimental approach for research on the problem-solving process]. In C. Comiti & G. Vergnaud (Eds.), *Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education* (pp. 278–283). Grenoble: Université de Grenoble, Laboratoire I.M.A.G.

Bell, A. W. (1976). *The learning of general mathematical strategies.* Unpublished doctoral dissertation, University of Nottingham.

Bloor, D. (1976). *Knowledge and social imagery.* London: Routledge & Kegan Paul.

Bruner, J. S., Goodnow, J. J., & Austin, G. A. (1956). *A study of thinking.* New York: Wiley.

Bruner, J. S., Olver, R. R., Greenfield, P. M., Hornsby, J. R., Kenney, H. J., Maccoby, M., Modiano, N., Mosher, F. A., Olson, D. R., Potter, M. C., Reich, L. C., & Sonstroem, A. M. (1966). *Studies in cognitive growth: A collaboration at the Center for Cognitive Studies.* New York: Wiley.

Burton, L. (1980a). Problems and puzzles. *For the Learning of Mathematics, 1*(2), 20–23.

Burton, L. (1980b). *The Skills and Procedures of Mathematical Problem Solving Project* (Social Science Research Council Project No. HR5410/1). London: Polytechnic of the South Bank.

Buxton, L. (1981). *Do you panic about maths? Coping with maths anxiety.* London: Heinemann.

de Bono, E. (1976). *Teaching thinking.* London: Temple Smith.

Edwards, J., & Baldauf, R. B. (1982). Teaching thinking in secondary science. In W. Maxwell (Ed.), *Thinking: The expanding frontier* (pp. 129–137). Philadelphia: Franklin Institute Press.

Floyd, A. (Ed.). (1982). *Developing mathematical thinking.* London: Addison-Wesley.

Gattegno, C. (1963). *For the teaching of mathematics* (Vol. 1). Reading, England: Educational Explorers.

Gattegno, C. (1973). *The universe of babies.* New York: Educational Solutions.

Gordon, W. J. J. (1961). *Synectics.* New York: Harper & Row.

Hofstadter, D. R. (1979). *Gödel, Escher, Bach: An eternal golden braid.* London: Harvester.

Lakatos, I. (1976). *Proofs and refutations.* Cambridge: Cambridge University Press.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically.* London: Addison-Wesley.

Parnes, S. J., & Noller, R. B. (1972–1973). Applied creativity: The Creative Studies Project. *Journal of Creative Behavior, 6,* 11–22, 164–186, 275–294; *7,* 15–36.

Schoenfeld, A. H. (1983). Episodes and executive decisions in mathematical problem solving. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 345–395). New York: Academic Press.

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